

PHYSICALLY PARAMETERIZED DIFFERENTIABLE MUSIC FOR DOA ESTIMATION WITH UNCALIBRATED ARRAYS

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MODEL-BASED MACHINE LEARNING

Typical data processing setting:

- We observe a **large** number of **correlated** variables, explained by a **small** number of **independent** factors.

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 - Large bias
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Hybrid approach: Model-based AI

Use models to structure, initialize and train learning methods

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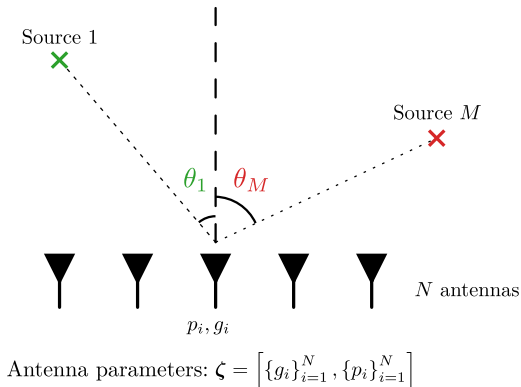
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- Make models more flexible: reduce bias of signal processing methods
- Guide machine learning methods: reduce their complexity

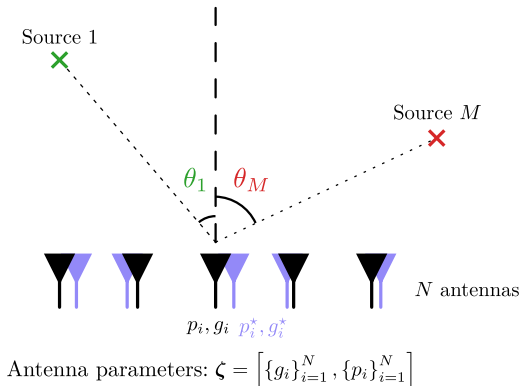
DOA ESTIMATION PROBLEM

- From measurements on N distinct antennas, how to estimate the direction of arrivals $\theta = [\theta_1, \dots, \theta_M]$ of M non-coherent far-field sources?



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DOA ESTIMATION PROBLEM: MATHEMATICAL FORMULATION

- System model:

$$\mathbf{X} = \mathbf{A}_\zeta(\boldsymbol{\theta}) \mathbf{S} + \mathbf{N} \quad (1)$$

with $\mathbf{X} \in \mathbb{C}^{N \times T}$, $\boldsymbol{\theta} \in [-\pi/2, \pi/2]^M$, $\mathbf{A}_\zeta(\boldsymbol{\theta}) \in \mathbb{C}^{N \times M}$, $\mathbf{S} \in \mathbb{C}^{M \times T}$, $\mathbf{N} \in \mathbb{C}^{N \times T}$

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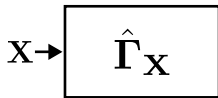
How to estimate θ from $\mathbf{X}$?

MUSIC METHOD

$\mathbf{X}$

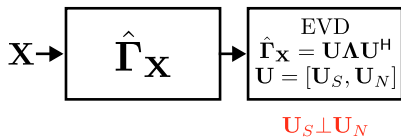
- Input: measurements

MUSIC METHOD



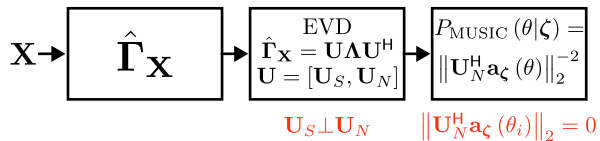
- Compute the sample covariance matrix from measurements

MUSIC METHOD



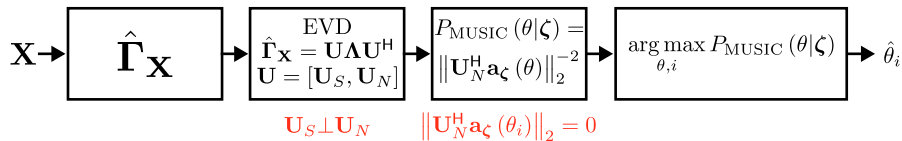
- Apply EVD on the sample covariance matrix

MUSIC METHOD



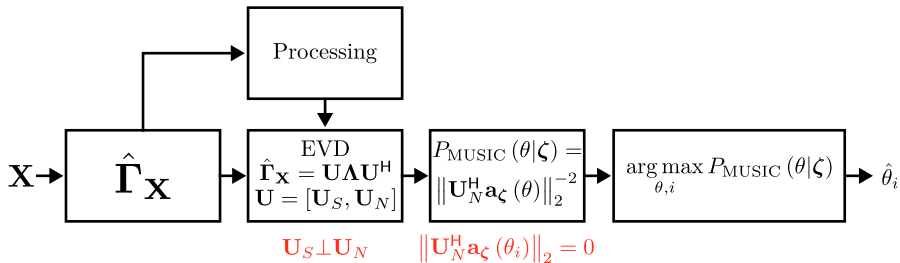
- Compute the MUSIC spectrum

MUSIC METHOD



- Find peaks and estimate DoAs

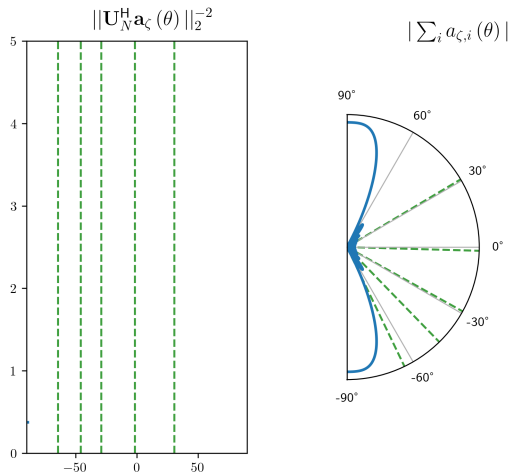
MUSIC METHOD



- If the sources are correlated: possibility of finding a surrogate covariance matrix through additional processing¹

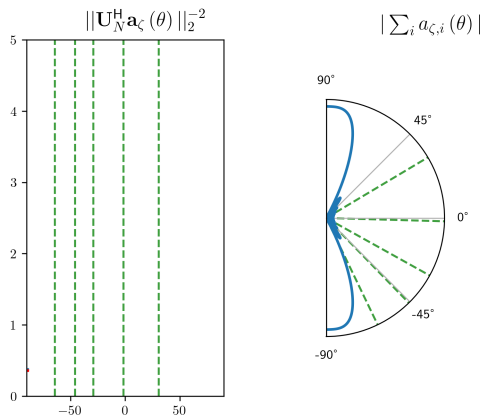
¹Shmuel et al., “SubspaceNet: Deep Learning-Aided Subspace Methods for DoA Estimation”.
CHATELIER et al. Physically Parameterized Differentiable MUSIC for DoA Estimation with Uncalibrated Arrays

MUSIC METHOD VISUALIZATION



- What happens if ζ is not perfectly known?

MUSIC METHOD VISUALIZATION



Estimation error if ζ is not perfectly known. How to learn ζ ?

CONTRIBUTIONS

- **Differentiable MUSIC algorithm to learn HWI through stochastic gradient-descent**

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- **Problem-specific supervised and unsupervised loss functions**

PROPOSED METHOD: MUSIC NON-DIFFERENTIABILITY

- Main idea: leverage SGD to solve

$$\begin{aligned} & \underset{\boldsymbol{\zeta}}{\text{minimize}} && \mathbb{E}_{(\boldsymbol{\theta}, \mathbf{X}) \sim \mathcal{P}_{(\boldsymbol{\theta}, \mathbf{X})}} \left[\mathcal{L} \left(\boldsymbol{\theta}, \hat{\boldsymbol{\theta}}(\mathbf{X} | \boldsymbol{\zeta}) \right) \right], \\ & \text{subject to} && \boldsymbol{\zeta} \in \mathbb{C}^N \times \mathbb{R}^N \end{aligned} \tag{P1}$$

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PROPOSED METHOD: MUSIC NON-DIFFERENTIABILITY

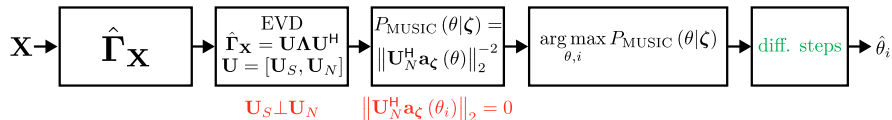
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 - The arg max in MUSIC leads to the non-existence of $\nabla_{\boldsymbol{\zeta}} \hat{\boldsymbol{\theta}}(\mathbf{X}|\boldsymbol{\zeta})$

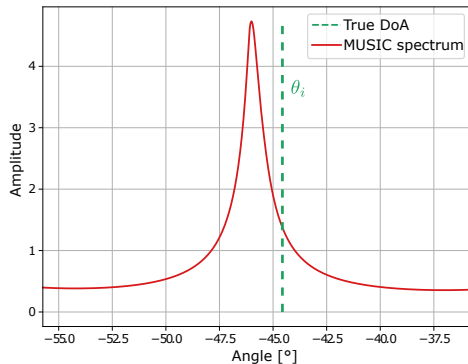
MUSIC is non-differentiable $\rightarrow$ diffMUSIC

PROPOSED METHOD: TOWARDS DIFFMUSIC



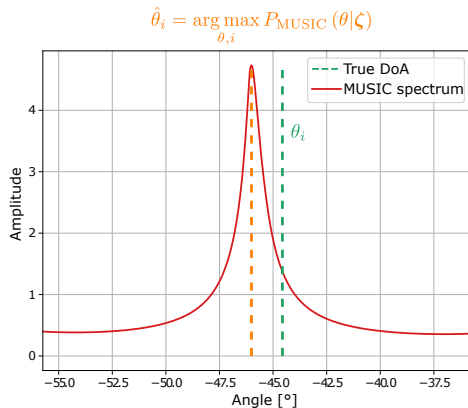
- diffMUSIC consists in the addition of differentiable processing steps after the peak-finding method.

PROPOSED METHOD: DIFFMUSIC DETAILS



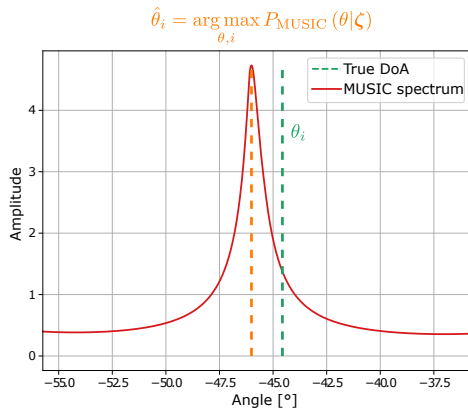
- Compute the MUSIC spectrum with current array knowledge ζ : $P_{\text{MUSIC}}(\theta|\zeta)$

PROPOSED METHOD: DIFFMUSIC DETAILS



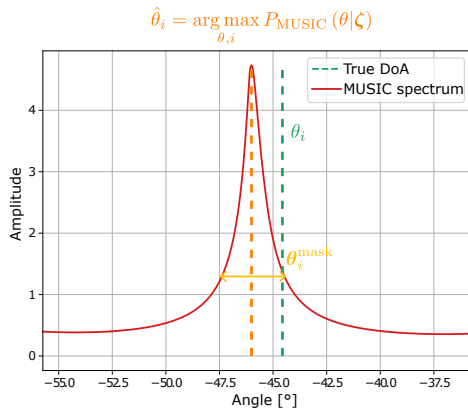
- Find peaks in the spectrum

PROPOSED METHOD: DIFFMUSIC DETAILS



- For each peak, estimate the DoA:

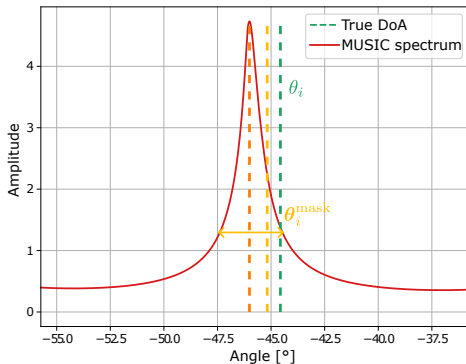
PROPOSED METHOD: DIFFMUSIC DETAILS



- For each peak, estimate the DoA:
 - Select neighbor angles through windowing: θ_i^{mask}

PROPOSED METHOD: DIFFMUSIC DETAILS

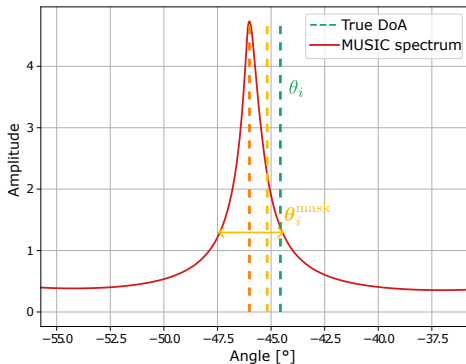
$$\hat{\theta}_i = \left(\theta_i^{\text{mask}}\right)^{\top} \text{softmax} \left(P_{\text{MUSIC}} \left(\theta_i^{\text{mask}} | \zeta \right) \right)$$



- For each peak, estimate the DoA:
 - Select neighbor angles through windowing: θ_i^{mask}
 - Convex combination: $\nabla_{\zeta} \hat{\theta}(\mathbf{X} | \zeta)$ exists $\rightarrow$ differentiable.

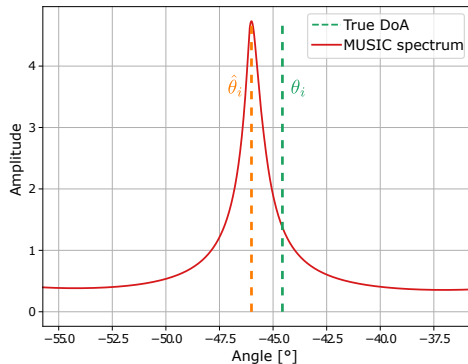
PROPOSED METHOD: DIFFMUSIC DETAILS

$$\hat{\theta}_i = \left(\theta_i^{\text{mask}}\right)^T \text{softmax} \left(P_{\text{MUSIC}} \left(\theta_i^{\text{mask}} | \zeta \right) \right)$$



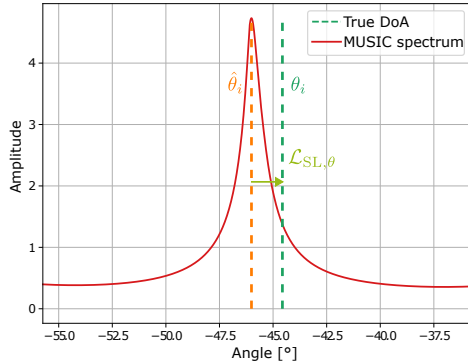
- Update the array parameters: $\zeta \leftarrow \zeta - \mu \nabla_{\zeta} \mathcal{L} \left(\theta, \hat{\theta}(\mathbf{X} | \zeta) \right)$

PROPOSED METHOD: LOSS FUNCTIONS



- How to design task-adapted loss functions to learn ζ ?

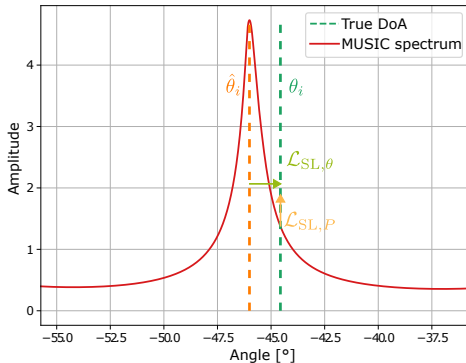
PROPOSED METHOD: LOSS FUNCTIONS



- Minimize the estimation error: RMSPE

$$\mathcal{L}_{SL,\theta} = \frac{1}{|\mathcal{T}|} \sum_{(\theta, \mathbf{X}) \in \mathcal{T}} \min_{\mathbf{P} \in \mathcal{P}} \sqrt{\frac{1}{M} \left\| \text{mod}_{\pi} \left(\boldsymbol{\theta} - \mathbf{P} \hat{\boldsymbol{\theta}}(\mathbf{X}|\boldsymbol{\zeta}) \right) \right\|_2^2} \quad (3)$$

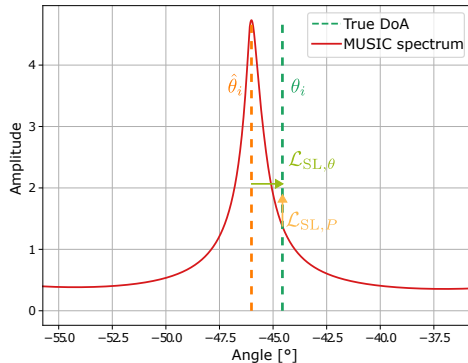
PROPOSED METHOD: LOSS FUNCTIONS



- Maximize spectrum amplitude at true DoA locations:

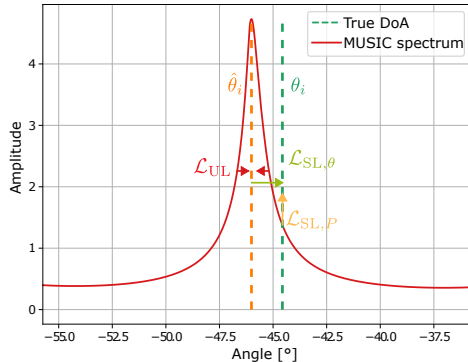
$$\mathcal{L}_{SL,P} = -\frac{1}{|\mathcal{T}|} \sum_{(\theta, \mathbf{X}) \in \mathcal{T}} \sum_i P_{\text{MUSIC}}(\theta_i | \zeta) \quad (3)$$

PROPOSED METHOD: LOSS FUNCTIONS



Requires *a-priori* knowledge of the true DoAs!

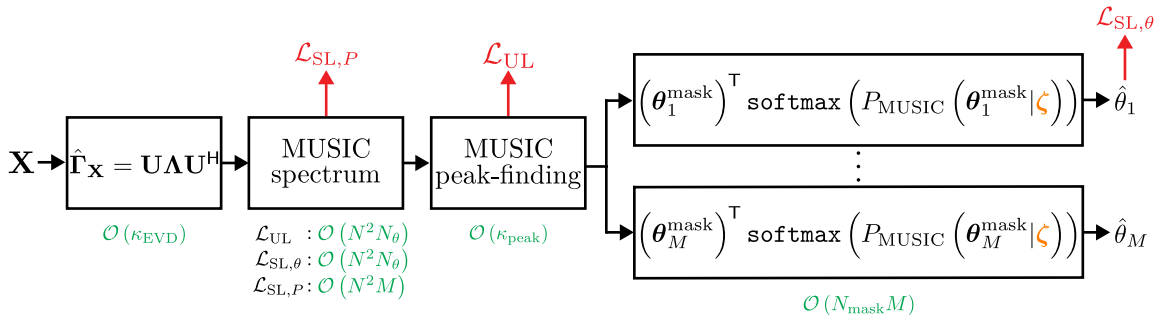
PROPOSED METHOD: LOSS FUNCTIONS



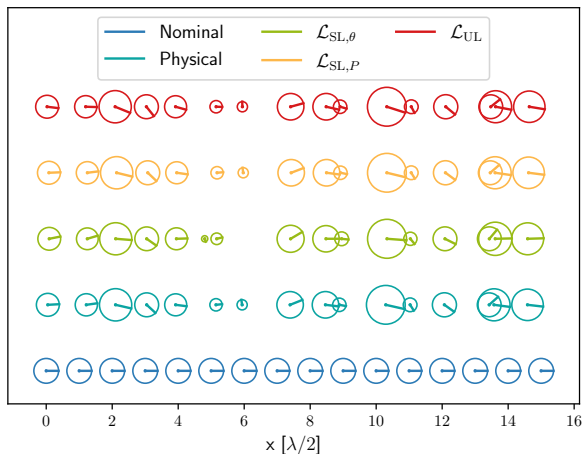
- **Unsupervised learning:** maximize spectrum sharpness within the chosen angular window (Jain's index based)

$$\mathcal{L}_{UL} = \frac{1}{|\mathcal{T}|} \sum_{\mathbf{X} \in \mathcal{T}} \sum_i \mathfrak{J} \left(P_{\text{MUSIC}} \left(\boldsymbol{\theta}_i^{\text{mask}} (\mathbf{X} | \boldsymbol{\zeta}) | \boldsymbol{\zeta} \right) \right) \quad (3)$$

PROPOSED METHOD: LOSS FUNCTIONS

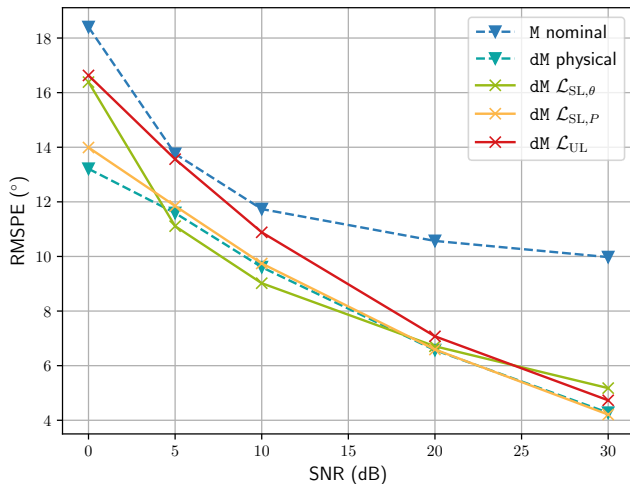


EXPERIMENTAL RESULTS: $N = 16, M = 5$, HIGH HWIS



- The proposed method learns the impairments

EXPERIMENTAL RESULTS: $N = 16, M = 5$, HIGH HWIS



- The proposed method performs well under noise

EXPERIMENTAL RESULTS: PERFORMANCE AGAINST BASELINES

		Baselines				$\mathcal{L}_{\text{SL},\theta}$		$\mathcal{L}_{\text{SL},P}$		$\mathcal{L}_{\text{UL}}$	
		M (nom.)	M (phys.)	dM (phys.)	SubspaceNet	M	dM	M	dM	M	dM
RMSPE ($^{\circ}$)	$M = 1$	2.425	0.014	0.013	0.098	0.019	0.015	0.013	0.013	1.339	1.310
	$M = 5$	9.976	4.358	4.275	16.123	5.371	5.178	4.325	4.209	4.834	4.731

Baseline comparisons

- The proposed method outperforms classical MUSIC and SubspaceNet

CONCLUSION

- Contributions:
 - MUSIC can be modified to be differentiable
 - HWIs can be learned while performing DoA estimation
 - Better results than MUSIC with unknown impairments

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- Contributions:
 - MUSIC can be modified to be differentiable
 - HWIs can be learned while performing DoA estimation
 - Better results than MUSIC with unknown impairments
- Future work:
 - Extend the method to coherent sources → spatial augmentation method
 - Extend the method to near-field → new dictionary expression
 - Combine with SubspaceNet → learn both the surrogate covariance matrix and the array parameters

THANKS!